Solution of singlet and non-singlet unpolarized DGLAP evolution equations in next-to-next-to-leading order (NNLO) by method of characteristics

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2 authors:

R. Baishya
J. N. College, Boko

Jayanta Sarma
Tezpur University, Napam, Tezpur, India

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ORIGINAL PAPER

Solution of singlet and non-singlet unpolarized DGLAP evolution equations in next-to-next-to-leading order (NNLO) by method of characteristics

R Baishya[1]* and J K Sarma[2]

[1]Department of Physics, Jawaharlal Nehru College, Boko 781 123, Assam, India

[2]Department of Physics, Tezpur University, Napaam 784 028, Assam, India

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Abstract: In this paper, the singlet and non-singlet structure functions have been obtained by solving Dokshitzer, Gribov, Lipatov, Altarelli, Parisi evolution equations in next-to- next-to-leading order at the small *x* limit. Here we have used Taylor Series expansion and then the method of characteristics to solve the evolution equation. We have also calculated *t* and *x*-evolutions of deuteron structure function as well as non singlet structure function and the results have been compared with the New Muon Collaboration, E665 experimental data, CLAS Collaboration and NNPDF Collaboration data.

Keywords: DIS; DGLAP equation; Small-x; Method of characteristics; Structure function; Splitting function; NNLO

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1. Introduction

The high-energy lepton-nucleon scattering has served as a sensitive probe for the substructure of the proton and neutron. Experiments with high energy electrons, muons and neutrinos have been used to characterize the parton substructure of the nucleon and to establish the current theory of the strong interaction, quantum chromodynamics etc. Here the observations are scaling violation for the unpolarized nucleon structure functions, the measurement of the strong coupling constant $\alpha_S(Q^2)$, the confirmation of numerous QCD sum rules, and the extraction of the parton distributions inside the nucleon. The solutions of the unpolarized DGLAP equation [1–4] for the Quantum Chromodynamics evolution of parton distribution functions have been discussed considerably over the past years [5–13]. There exist two main classes of approaches: those that solve the equation directly in x-space and those that solve it for Mellin transforms of the parton densities and subsequently invert the transform back to *x*-space. Some available programs that deal with the DGLAP evolution are PEGASUS [14], based on the use of Mellin moments and QCDNUM [15], CANDIA [16] and HOPPET [17], all of which are based on *x*- space.

The computation of the three-loop contributions to the anomalous dimensions is needed to complete the next to next to leading order (NNLO) calculations for DIS. The NNLO corrections should be included in order to arrive at quantitatively reliable predictions for hard processes at present and future high energy colliders. Recently the three loop splitting functions are introduced with a good phenomenological success [11, 12, 18–23]. Though various methods are available in order to obtain a numerical solution of DGLAP evolution equations, exact analytical solutions are not known. Here we solve the DGLAP evolution equation in NNLO analytically by using method of characteristics and get unique solution having good agreement with recent experimental data. Hence it is significant as an important phenomenological work for studying structure functions.

2. Theory

The DGLAP evolution equation [1–4] in the standard form is given by

*Corresponding author, E-mail: rjitboko@yahoo.co.in

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$$\frac{\partial}{\partial \ln Q^2}\begin{pmatrix} q_S \\ g \end{pmatrix} = \begin{pmatrix} P_{qq} & P_{qg,} \\ P_{gq} & P_{gg} \end{pmatrix} \otimes \begin{pmatrix} q_S \\ g \end{pmatrix}, \quad (1)$$

For evolution of singlet structure function we have to calculate the quark–quark splitting function P_{qq} and gluon-quark splitting function P_{qg}. But for evolution of non-singlet structure function we have to calculate only the quark–quark splitting function P_{qq}. One can write

$$P_{qq}(x,Q^2) = \frac{\alpha_S(Q^2)}{2\pi} P^{(0)}_{qq}(x) + \left(\frac{\alpha_S(Q^2)}{2\pi}\right)^2 P^{(1)}_{qq}(x) + \left(\frac{\alpha_S(Q^2)}{2\pi}\right)^3 P^{(2)}_{qq}(x), \quad (2)$$

where $P^{(0)}_{qq}(x)$, $P^{1}_{qq}(x)$ and $P^{(2)}_{qq}(x)$ are LO, NLO and NNLO splitting functions respectively. Splitting functions $P^{(0)}_{qq}(x)$ and $P^{1}_{qq}(x)$ are already defined and using these, the DGLAP equation has been solved up to NLO in our previous papers [19, 24–27]. By adding $P^{(2)}_{qq}(x)$ with previous terms we will get the NNLO evolution equation. The three loop quark–quark splitting function P_{qq} can be expressed as

$$P_{qq} = P^{(2)}_{NS} + P^{(2)}_{PS} \quad (3)$$

The non singlet contribution $P^{(2)}_{NS}$ [11, 12, 22] dominates P_{qq} at large *x*, where the 'pure singlet' term $P^{(2)}_{PS}$ [23] is very small. At small *x*, on the other hand, the latter contribution takes over as $xP^{(2)}_{PS}$ does not vanish for $x \to 0$, unlike $xP^{(2)}_{NS}$. The non-singlet splitting functions $P^{(2)}_{NS}(x)$ can be obtained from the *N*- space results of the Mellin space by an inverse Mellin transformation.

After simplification, the singlet and non singlet DGLAP evolution equations in NNLO take the following form

$$\frac{\partial F^S_2}{\partial t} - \frac{\alpha_S(t)}{2\pi}\left[\frac{2}{3}\{3+4\ln(1-x)\}F^S_2(x,t) + I^S_1(x,t)\right] - \left(\frac{\alpha_S(t)}{2\pi}\right)^2 I^S_2(x,t) - \left(\frac{\alpha_S(t)}{2\pi}\right)^3 I^S_3(x,t) = 0 \quad (4)$$

$$\frac{\partial F^{NS}_2}{\partial t} - \frac{\alpha_S(t)}{2\pi}\left[\frac{2}{3}\{3+4\ln(1-x)\}F^{NS}_2(x,t) + I^{NS}_1(x,t)\right] - \left(\frac{\alpha_S(t)}{2\pi}\right)^2 I^{NS}_2(x,t) - \left(\frac{\alpha_S(t)}{2\pi}\right)^3 I^{NS}_3(x,t) = 0 \quad (5)$$

where $I^S_1(x,t)$, $I^S_2(x,t)$, $I^S_3(x,t)$, $I^{NS}_1(x,t)$, $I^{NS}_2(x,t)$ and $I^{NS}_3(x,t)$ are integral functions.

Now let us introduce the variable $u = 1-\omega$ and since $x < \omega < 1$, so $0 < u < 1-x$, and hence the series is convergent for $|u| < 1$. As x is small in our region of discussion, using Taylor's expansion series, we can rewrite

$$F^S_2\left(\frac{x}{\omega},t\right) = F^S_2(x,t) + \frac{xu}{1-u}\frac{\partial F^S_2(x,t)}{\partial x}, \quad (6)$$

$$G\left(\frac{x}{\omega},t\right) = G(x,t) + \frac{xu}{1-u}\frac{\partial G(x,t)}{\partial x} \quad (7)$$

Using these Equations and performing *u*-integrations, Eq. (4) takes the form

$$\frac{\partial F^S_2(x,t)}{\partial t} - \frac{\alpha_s}{2\pi}\left[A_1(x)F^S_2(x,t) + A_2(x)\frac{\partial F^S_2(x,t)}{\partial x} + A_3(x)G(x,t) + A_4(x)\frac{\partial G(x,t)}{\partial x}\right]$$
$$-\left(\frac{\alpha_s}{2\pi}\right)^2\left[B_1(x)F^S_2(x,t) + B_2(x)\frac{\partial F^S_2(x,t)}{\partial x} + B_3(x)G(x,t) + B_4(x)\frac{\partial G(x,t)}{\partial x}\right]$$
$$-\left(\frac{\alpha_s}{2\pi}\right)^3\left[C_1(x)F^S_2(x,t) + C_2(x)\frac{\partial F^S_2(x,t)}{\partial x} + C_3(x)G(x,t) + C_4(x)\frac{\partial G(x,t)}{\partial x}\right] = 0 \quad (8)$$

where $A_1(x)$, $A_2(x)$, $A_3(x)$, $A_4(x)$, $B_1(x)$, $B_2(x)$, $B_3(x)$, $B_4(x)$, $C_1(x)$, $C_2(x)$, $C_3(x)$ and $C_4(x)$ are some functions of *x*.

In order to solve Eq. (8), we need to relate the singlet distribution function $F^S_2(x,t)$ with the gluon distribution function $G(x,t)$. For small *x* and high Q^2, the gluon is expected to be more dominant than the sea quark. But for lower Q^2, there is no such clear-cut distinction between the two. Hence, for simplicity, let us assume $G(x,t) = k(x)F^S_2(x,t)$, where *k*(*x*) is a suitable function of *x* or may be a constant. Also we have considered two numerical parameters T_0 and T_1, such that $T^2(t) = T_0 \cdot T(t)$ and $T^3(t) = T_0 \cdot T(t) \cdot T(t) = T_1 \cdot T(t)$, where $T(t) = \frac{\alpha_S(t)}{2\pi}$ [24–26]. Thus Eq. (8) takes the form

$$-t\frac{\partial F^S_2(x,t)}{\partial t} + L(x)\frac{\partial F^S_2(x,t)}{\partial x} + M(x)F^S_2(x,t) = 0, \quad (9)$$

where

$$L(x) = A_f[(A_2 + KA_4) + T_0(B_2 + KC_4) + T_1(C_2 + KC_4)],$$

$$M(x) = A_f\left[\left(A_1 + KA_3 + \frac{\partial K}{\partial x}A_4\right) + T_0\left(B_1 + KB_3 + \frac{\partial K}{\partial x}B_4\right) + T_1\left(C_1 + KC_3 + \frac{\partial K}{\partial x}C_4\right)\right]$$

To introduce the method of characteristics, let us consider two new variables S and τ instead of *x* and *t*, such that $\frac{dt}{dS} = -t$ and $\frac{dx}{dS} = L(x)$, which are known as characteristic equations. Putting these in Eq. (9), we get $\frac{dF_2^S(S,\tau)}{dS} + M(S,\tau)F_2^S(S,\tau) = 0$, which can be solved as

$$F_2^S(S,\tau) = F_2^S(0,\tau)\exp\left[-\int_0^S M(S,\tau)dS\right] \quad (10)$$

For initial condition $S = 0 \Rightarrow t = t_0$ and $F_2^S(S,\tau) = F_2^S(0,\tau)$. Now we have to replace the co-ordinate system (S, τ) by (x, t) with the input function $F_2^S(0,\tau) = F_2^S(x,t_0)$ and get the *t*-evolution of singlet structure function in NNLO as

$$F_2^S(x,t) = F_2^S(x,t_0)\left(\frac{t}{t_0}\right)^{A_f[P_1+T_0P_2+T_1P_3]} \quad (11)$$

Similarly the x-evolution of singlet structure function in NNLO will be

$$F_2^S(x,t) = F_2^S(x_0,t)\int_{x_0}^{x}\frac{[P_1+T_0P_2+T_1P_3]}{[Q_1+T_0Q_2+T_1Q_3]}dx \quad (12)$$

where

$$P_1 = \left(A_1 + K(x)A_3 + \frac{\partial K(x)}{\partial x}A_4\right),$$
$$P_2 = \left(B_1 + K(x)B_3 + \frac{\partial K(x)}{\partial x}B_4\right),$$
$$P_3 = \left(C_1 + K(x)C_3 + \frac{\partial K(x)}{\partial x}C_4\right),$$
$$Q_1 = (A_2 + K(x)A_4),$$
$$Q_2 = (B_2 + K(x)C_4),$$
$$Q_3 = (C_2 + K(x)C_4)$$

Thus the *t* and *x*-evolution of deuteron structure functions in NNLO can be obtained as

$$F_2^d(x,t) = F_2^d(x,t_0)\left(\frac{t}{t_0}\right)^{A_f[P_1+T_0P_2+T_1P_3]} \quad (13)$$

and

$$F_2^d(x,t) = F_2^d(x_0,t)\int_{x_0}^{x}\frac{[P_1+T_0P_2+T_1P_3]}{[Q_1+T_0Q_2+T_1Q_3]}dx, \quad (14)$$

where $F_2^d(x,t_0) = \frac{5}{2}F_2^S(x,t_0)$ and $F_2^d(x_0,t) = \frac{5}{2}F_2^s(x_0,t)$ are input functions.

Proceeding in the same way we will get the t and x evolution of non singlet structure function from Eq. (5) as

$$F_2^{NS}(x,t) = F_2^{NS}(x,t_0)\left(\frac{t}{t_0}\right)^{A_f[A_1+T_0B_1+T_1C_1]} \quad (15)$$

and

$$F_2^{NS}(x,t) = F_2^{NS}(x_0,t)\int_{x_0}^{x}\frac{[A_1+T_0B_1+T_1C_1]}{[A_2+T_0B_2+T_1C_2]}dx, \quad (16)$$

To compare our results with the experimental data, we have to consider the relation between proton and deuteron structure functions measured in DIS with non-singlet quark distribution function as $F_2^{NS}(x,t) = 3\left[2F_2^p(x,t) - F_2^d(x,t)\right]$.

3. Results and discussions

Our results of Eq. (13) for *t*-evolution and Eq. (14) for *x*-evolution of deuteron structure function $F_2^d(x,t)$ are compared with NMC data [28] (in muon- deuteron DIS with incident momentum 90, 120, 200, 280 GeV^2), CLAS Collaboration [29] as well as NNPDF Collaboration [30] data. We have also compared our results of Eqs. (15) and (16) for *t* and *x*-evolutions of non-singlet structure function $F_2^{NS}(x,t)$ with NMC and E-665 [31] experimental data.

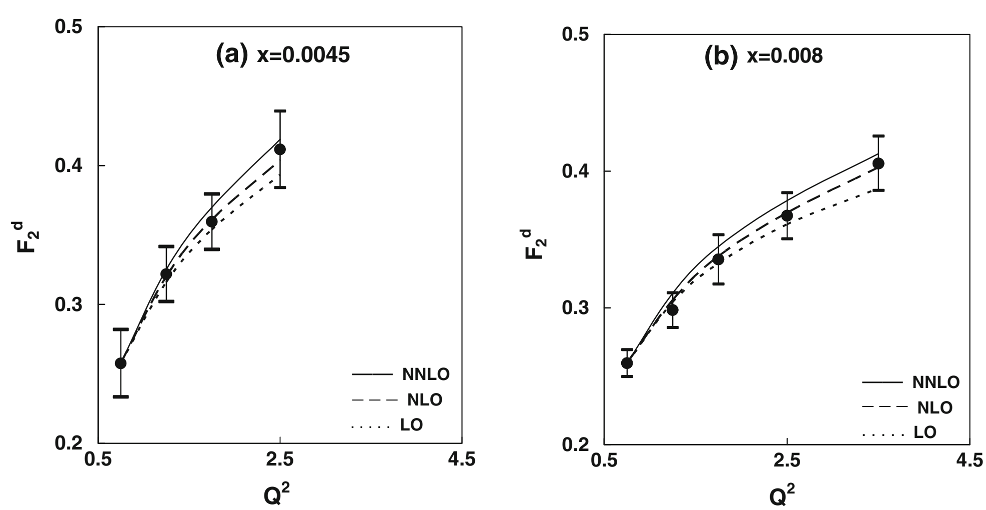

Fig. 1 *t*-evolution of deuteron structure functions compared with NMC data. $F_2^d(x,t)$ are plotted against Q^2 keeping *x* constant for **a** $x = 0.0045$ and **b** $x = 0.008$

We have considered the range $0.4 \leq Q^2 \leq 6.0$ GeV^2 for CLAS Collaboration, $5.0 \leq Q^2 \leq 50.0$ GeV^2 for NNPDF Collaboration, $0.01 \leq x \leq 0.0489$ and $1.496 \leq Q^2 \leq 13.391$ GeV^2 for E665 data also $0.0045 \leq x \leq 0.14$ and $0.75 \leq Q^2 \leq 27$ GeV^2 for NMC data. It is observed that, within these ranges, for the minimum error we have to consider $T_0 = 0.048$ and $T_1 = 0.003$ [19, 24–26].

In Fig. 1, t-evolutions have been plotted as $F_2^d(x,t)$ against Q^2 keeping x constant having the values of 0.0045, 0.008, 0.0125 and 0.0175 respectively. NNLO results are compared with our NLO, LO results and NMC data. In Fig. 2, x-evolutions have been plotted as $F_2^d(x,t)$ against x keeping Q^2 constant with the values of $Q^2 = 11.5$ GeV^2, 15.0 GeV^2, 20.0 GeV^2 and 27.0 GeV^2 respectively and our NNLO results are compared with NMC data. Also we compared our NNLO results with our NLO and LO results.

In Fig. 3, we have plotted computing values of $F_2^d(x,t)$ for NNLO, against the x values for a fixed Q^2 with $k(x) = k$, a constant and our results are compared with CLAS Collaboration data. Also we compared NNLO results with our NLO and LO results. Though the DGLAP evolution equation are satisfied at high Q^2 and small-x, but

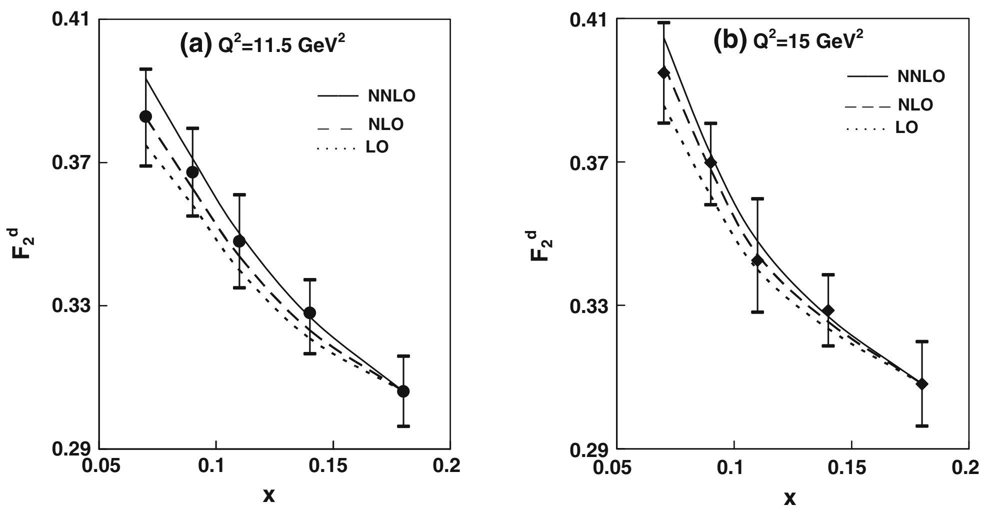

Fig. 2 x-evolution of deuteron structure function compared with NMC data. $F_2^d(x,t)$ are plotted against x keeping Q^2 constant for **a** $Q^2 = 11.5$ GeV^2 and **b** $Q^2 = 15.0$ GeV^2

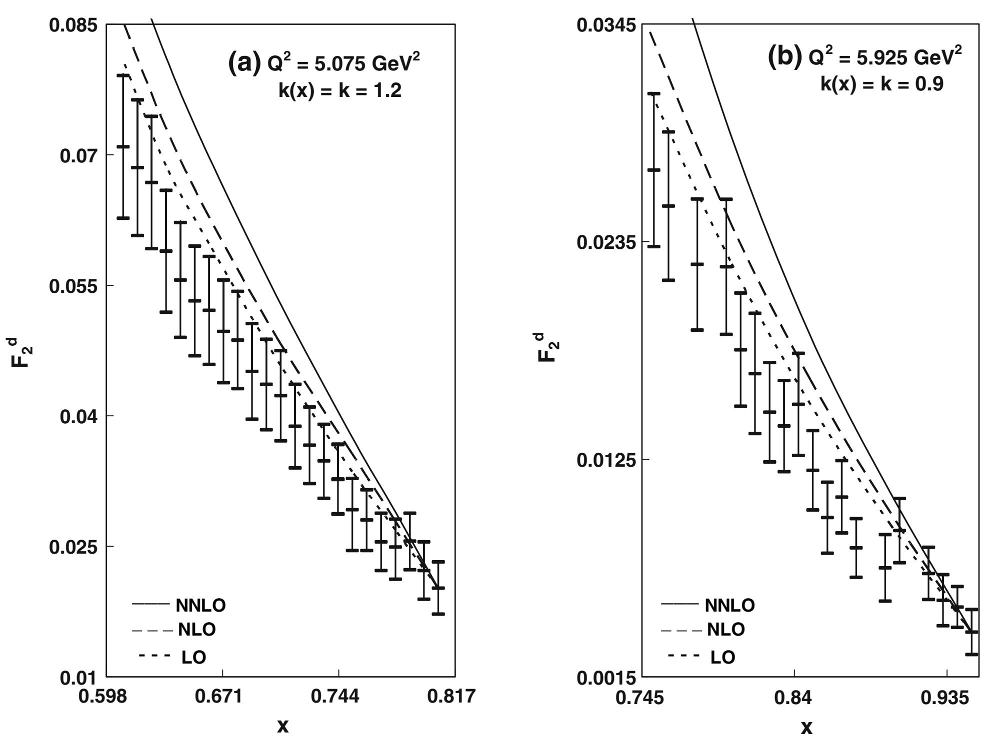

Fig. 3 x-evolution of deuteron structure functions compared with CLAS Collaboration data. $F_2^d(x,t)$ are plotted against x keeping Q^2 constant for **a** $Q^2 = 5.075$ GeV^2 and **b** $Q^2 = 5.925$ GeV^2

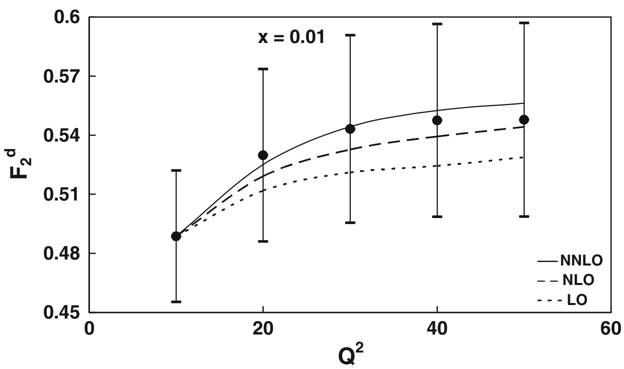

Fig. 4 *t*-evolution of deuteron structure functions in NNLO compared with NNPDF Collaboration data considering $k\ (x) = k$, a constant

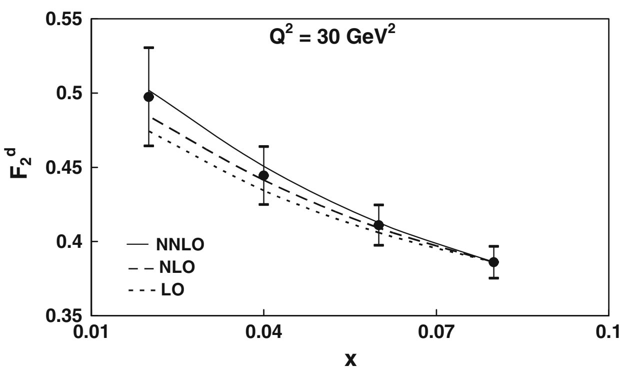

Fig. 5 *x*-evolution of deuteron structure functions in NNLO compared with NNPDF Collaboration data considering $k(x) = k$, a constant

CLAS data are available at comparably smaller Q^2 and higher x. Thus our NNLO results are rather satisfied with CLAS Collaboration data. The nearly fitting curves with high Q^2 and small-x ranges that available in CLAS Collaboration get for $0.6 \leq k \leq 1.0$.

In Figs. 4 and 5, we have plotted our computed NNLO values of $F_2^d(x,t)$ against the Q^2 values in a fixed x for t-evolution and against the x values in a fixed Q^2 for x-evolution and our results are compared with NNPDF collaboration data. Also we have compared the NNLO results with our NLO and LO results. Here we have considered $k(x) = k$, a constant and best-fit curves get for $k = 1.1$ at $x = 0.01$ and $Q^2 = 30\ \text{GeV}^2$.

In Fig. 6, t-evolutions have been plotted as $F_2^{NS}(x,t)$ against Q^2 keeping x constant with the values of 0.01, 0.017, 0.024 and 0.035 for E665 data and 0.0045, 0.008, 0.0125 and 0.0175 for NMC data. Here also we have compared the NNLO results with our NLO and LO results. For clarity, data are scaled up by +0.5i for E-665 data and by +0.3i (i = 0, 1, 2, 3, - -) for NMC data, starting from bottom of all graphs in each Figure.

In Fig. 7, x-evolutions have been plotted as $F_2^{NS}(x,t)$ against x keeping Q^2 constant with the values of $Q^2 = 5.236\ \text{GeV}^2, 7.176\ \text{GeV}^2, 9.795\ \text{GeV}^2, 13.391\ \text{GeV}^2$ for E665 data and $9.0\ \text{GeV}^2$, $11.5\ \text{GeV}^2$, $15.0\ \text{GeV}^2$, $20.0\ \text{GeV}^2$ for NMC data. Here also we have compared the NNLO results with our NLO and LO results. For clarity, data are scaled up by +0.5i for E665 data and by +0.2i for NMC data, starting from bottom of all graphs in each Figure.

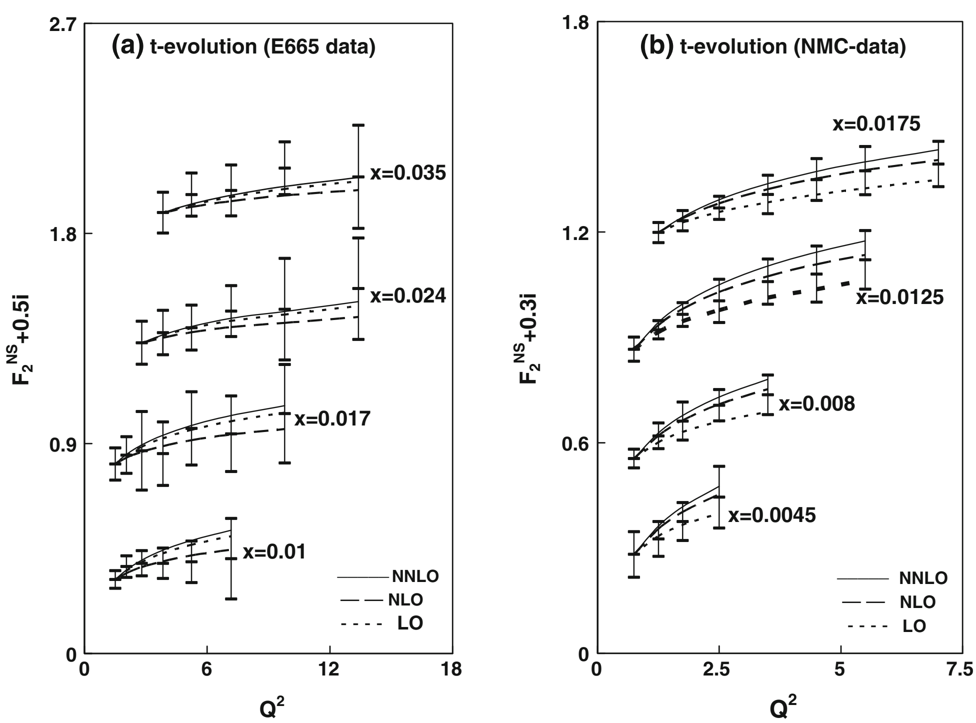

Fig. 6 *t*-evolution of non-singlet structure function $F_2^{NS}(x,t)$ are plotted against Q^2 keeping x constant compared with **a** E—665 data and **b** NMC data

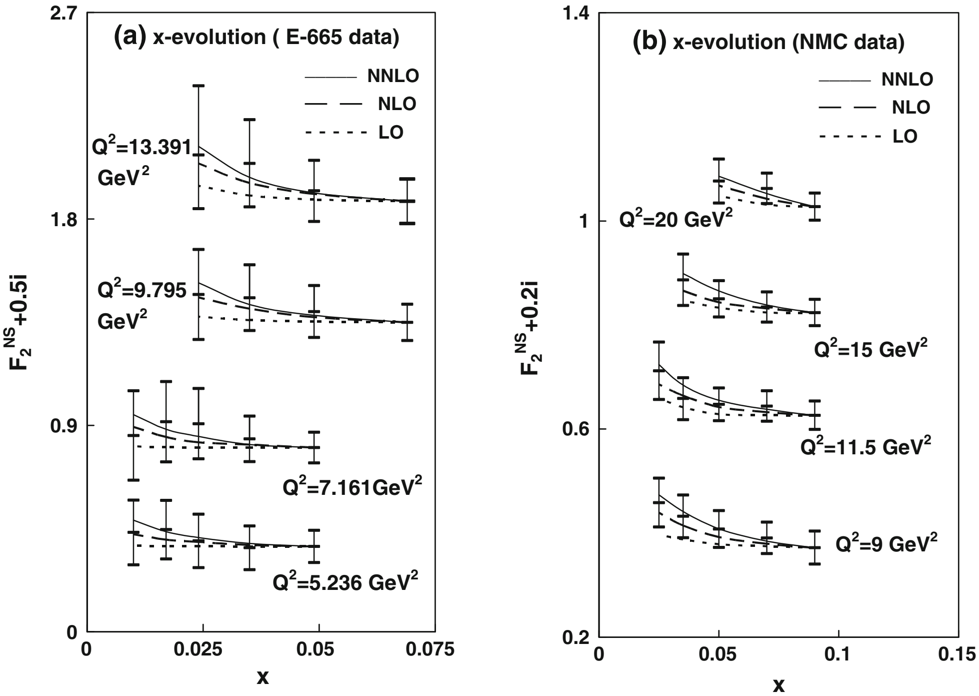

Fig. 7 *x*-evolution of non singlet structure functions. $F_2^{NS}(x,t)$ are plotted against *x* keeping Q^2 constant compared with **a** E—665 data and **b** NMC data

4. Conclusions

In this paper, we have solved the DGLAP evolution equation by the method of characteristics and have obtained the singlet and non-singlet structure functions. Here we have found that the *t* and *x*-evolution of deuteron structure function as well as non-singlet structure function, which is the combination of proton and deuteron structure functions, are in good consistency with the NMC, E-665, CLAS Collaboration data sets and NNPDF collaboration parameterization results. On an average the percentage errors of our results in LO, NLO and NNLO are 0.0067, 0.0049 and 0.0036 % with NMC data, 0.0054, 0.0032 and 0.0016 % with E-665 data, 0.0068, 0.0043 and 0.0029 % with CLAS data also 0.0036, 0.0083 and 0.014 % with NNPDF Collaboration results respectively. Here except in case of NNPDF Collaboration, the contribution of NNLO is found to be high at the lower-*x* and higher Q^2. In NNPDF Collaboration, data are available only at lower Q^2 and higher–*x* where DGLAP equation does not hold good. Here we can claim that in our presentation, we consider very few numbers of parameters in comparison to the other methods.

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