Solution of non-singlet DGLAP evolution equation in leading order and next-to-leading order at small-x by method of characteristics

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Abstract : The non-singlet structure functions have been obtained by solving Dokshitzer, Gribove, Lipatov, Alterelli, Parisi (DGLAP) evolution equations in leading order (LO) and next to leading order (NLO) at the small x limit. Here a Taylor Series expansion has been used and then the method of characteristics has been used to solve the evolution equations. We have also calculated t and x-evolutions of deuteron structure function and the results are compared with the New Muon Collaboration (NMC) and E665 data.

Keywords : DIS; DGLAP equation; small-x; method of characteristics; structure function.

PACS Nos. : 12.35 Eq.; 12.38-t; 12.39-x; 13.60 Hb.

1. Introduction

Structure functions in deep-inelastic lepton-hadron scattering (DIS) remain among the most important probes of perturbative Quantum Chromodynamics (PQCD) and of the partonic structure of hadrons. Indeed, experiments have proceeded towards very high accuracy and a greatly extended kinematic coverage during the past two decades. An accurate knowledge of the parton densities will be indispensable for interpreting many results at the future for hadrons structure. The non-perturbative Bjorken-x dependence of the structure functions at one scale, the scaling violations can be calculated in the QCD-improved parton model in terms of a power series expansion in the strong coupling constant α_s. The next-to-leading order (NLO) ingredients for such analyses are available since 1980 for unpolarized structure functions in mass less perturbative QCD [1]. The corresponding

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results for the next-to-next-to leading order (NNLO) are complete at present, though there is enormous complexity in the required loop calculations [2-6]. One of the most useful and famous evolution equation is the Dokshitzer, Gribov, Lipatov, Alterelli, Parisi (DGLAP) equation [7-14] to give $t\left[=\ln\left(Q^2/\Lambda^2\right), \Lambda\right.$, is the QCD cut off parameter] and x evolutions of structure functions. Hence the solutions of DGLAP evolution equations give quark and gluon structure functions that produce ultimately proton, neutron and deuteron structure functions.

Though various methods are available in order to obtain a numerical solution of DGLAP evolution equations, exact analytical solutions of these equations are not known [5-6]. Various analytical methods have been reported to solve DGLAP evolution equations and one of the limitations of these solutions is that, as they are partial differential equations (PDE), their ordinary solutions are not unique one. On the other hand, this limitation can be overcome by using method of characteristics [15-18]. The singlet structure functions at small-x in LO and NLO have been obtained from DGLAP equation by using the method of characteristics in the Ref. [18]. In this paper, we obtain a solution of DGLAP equations for non-singlet structure functions at small-x in LO and NLO by using the method of characteristics. The results are compared with E665 and NMC data [19, 20].

2. Theory

The non singlet combinations of quark and anti quark densities, q_i and q_i^-, are given by

$$q_{NS}^{\pm} = q_i \pm q_i^- - \left(q_k \pm q_k^-\right), q_{NS}^V = \sum_{r=1}^{N_f}\left(q_r - q_r^-\right) \tag{1}$$

N_f stands for the number of effectively massless flavours. The corresponding splitting functions are denoted by $P_{NS}^{\pm}$ and $P_{NS}^V = P_{NS}^- + P_{NS}^S$. For an integro-differential equation of DGLAP type, which is in a perturbative fashion, the kernel $P(x)$ is known perturbatively up to the first few orders in α_S, approximations which are commonly known as LO, NLO, NNLO. The evolution equation in the non singlet sector [21-23] is

$$\frac{\partial F_2^{NS}\left(x,Q^2\right)}{\partial \ln Q^2} = P_{NS}\left(x,Q^2\right) \otimes F_2^{NS}\left(x,Q^2\right) \tag{2}$$

Here $\otimes$ stand for the Mellin convolution in the momentum variable [4, 6] and define as

$$a(x) \otimes b(x) \equiv \int_0^1 \frac{dy}{y} a(y)\, b\left(\frac{x}{y}\right) \tag{3}$$

Also

$$P\left(x,Q^2\right) = \frac{\alpha_S\left(Q^2\right)}{2\pi}P^{(0)}\left(x\right) + \left(\frac{\alpha_S\left(Q^2\right)}{2\pi}\right)^2 P^{(1)}\left(x\right) + \cdots \tag{4}$$

Here $P^{(0)}(x)$, $P^{(1)}(x)$ are splitting functions [24] in LO, NLO respectively. The DGLAP evolution equations for non-singlet structure functions in LO and NLO [1, 21, 22] can be written as

$$\frac{\partial F_2^{NS}}{\partial t} - \frac{\alpha_S(t)}{2\pi}\left[\frac{2}{3}\{3 + 4\ln(1-x)\} F_2^{NS}(x,t) + I_1^{NS}(x,t)\right] = 0 \tag{5}$$

$$\frac{\partial F_2^{NS}}{\partial t} - \frac{A_f}{t}\left[\{3 + 4\ln(1-x)\} F_2^{NS}(x,t) + I_1^{NS}(x,t)\right] - \left(\frac{\alpha_S(t)}{2\pi}\right)^2 I_2^{NS} = 0 \tag{6}$$

where

$$I_1^{NS}(x,t) = \frac{4}{3}\int_x^1 \frac{d\omega}{1-\omega}\left[\left(1+\omega^2\right)F_2^{NS}\left(\frac{x}{\omega},t\right) - 2F_2^{NS}(x,t)\right], \tag{7a}$$

$$I_2^{NS} = \left[(x-1)F_2^{NS}(x,t)\int_0^1 f(\omega)\,d\omega + \int_x^1 f(\omega)\,F_2^{NS}\left(\frac{x}{\omega},t\right)d\omega\right]. \tag{7b}$$

$$f(\omega) = C_F^2\left[P_F(\omega) - P_A(\omega)\right] + \frac{1}{2}C_F C_A\left[P_G + P_A(\omega)\right] + C_F T_R N_f P_{N_f}(\omega),$$

$$P_F(\omega) = -\frac{2\left(1+\omega^2\right)}{(1-\omega)}\ln(\omega)\ln(1-\omega) - \left(\frac{3}{1-\omega} + 2\omega\right)\ln\omega - \frac{1}{2}(1+\omega)\ln\omega + \frac{40}{3}(1-\omega),$$

$$P_G(\omega) = \frac{\left(1+\omega^2\right)}{(1-\omega)}\left(\ln^2(\omega) + \frac{11}{3}\ln(\omega) + \frac{67}{9} - \frac{\pi^2}{3}\right) + 2(1+\omega)\ln\omega + \frac{40}{3}(1-\omega),$$

$$P_{N_f}(\omega) = \frac{2}{3}\left[\frac{1+\omega^2}{1-\omega}\left(-\ln\omega - \frac{5}{3}\right) - 2(1-\omega)\right],$$

$$P_A(\omega) = \frac{2\left(1+\omega^2\right)}{(1+\omega)}\int_{\left(\frac{\omega}{1+\omega}\right)}^{\left(\frac{1}{1+\omega}\right)} \frac{dk}{k}\ln\left(\frac{1-k}{k}\right) + 2(1+\omega)\ln(\omega) + 4(1-\omega),$$

with $C_A = C_G = 3,\ C_F = \frac{\left(N_f^2 - 1\right)}{2N_f}, T_f = \frac{1}{2},\ \alpha_S(t) = \frac{4\pi}{\beta_0 t}\left[1 - \frac{\beta_1 \ln t}{\beta_0^2 t}\right].$

$\beta_0 = 11 - \frac{2}{3}N_f$ and $\beta_1 = 102 - \frac{38}{3}N_f$ are the one loop (LO) and two loop (NLO) correction to the QCD β- function and N_f being the flavors number. We can neglect β_1 for LO.

Let us introduce the variable $u = 1 - \omega$ and since $x < \omega < 1$, so $0 < u < 1 - x$, hence the series is convergent for $|u| < 1$.

Now $\frac{x}{\omega} = \frac{x}{1-u} = \left(x + \frac{xu}{1-u}\right).$

So, using Taylor's expansion series we can rewrite

$$F_2^{NS}\left(\frac{x}{\omega}, t\right) = F_2^{NS}\left(x + \frac{xu}{1-u}, t\right)$$

$$= F_2^{NS}(x,t) + \frac{xu}{1-u}\frac{\partial F_2^{NS}(x,t)}{\partial x} + \frac{1}{2}\left(\frac{xu}{1-u}\right)^2 \frac{\partial^2 F_2^{NS}(x,t)}{\partial x^2} + - - - -$$

$$F_2^{NS}\left(\frac{x}{\omega}, t\right) = F_2^{NS}(x,t) + \frac{xu}{1-u}\frac{\partial F_2^{NS}(x,t)}{\partial x} \quad (8)$$

Since x is small in our region of discussion, the terms containing x^2 and higher powers of x can be neglected. Using eq. (8) in eq. (7a) and (7b) and performing u-integrations, eq. (5) becomes the form

$$-t\frac{\partial F_2^{NS}(x,t)}{\partial t} + A_f A(x) F_2^{NS}(x,t) + A_f B(x)\frac{\partial F_2^{NS}}{\partial x} = 0 \quad (9)$$

where

$$A_f = \frac{\alpha_S(t)}{3\pi}t = \frac{4}{3\beta_0} = \frac{4}{33 - 2N_f}$$

$$A(x) = 2x + x^2 + 4\ln(1-x), \quad (10a)$$

$$B(x) = x - x^3 - 2x\ \ln(x), \quad (10b)$$

To introduce method of characteristics, let us consider two new variables S and τ instead of x and t, such that

$$\frac{dt}{dS} = -t \, , \tag{11a}$$

$$\frac{dx}{dS} = A_f B(x) \tag{11b}$$

which are known as characteristics eqs. [15-18]. Putting these equations in (9), we get

$$\frac{dF_2^{NS}(S,\tau)}{dS} + L(S,\tau) F_2^{NS}(S,\tau) = 0 \, , \tag{12}$$

which can be solved as-

$$F_2^{NS}(S,\tau) = F_2^{NS}(\tau)\left(\frac{t}{t_0}\right)^{L(S,\tau)} , \tag{13}$$

where $L(S,\tau) = 3/2 \, A_f \, A(x)$ and $F_2^{NS}(S,\tau) = F_2^{NS}(\tau)$ for initial condition S = 0, t = t_0.

Now we have to replace the co-ordinate system (S,τ) to (x, t) with the input function $F_2^{NS}(\tau) = F_2^{NS}(x,t_0)$ and will get the t- evolution of non singlet structure function in the LO as

$$F_2^{NS}(x,t) = F_2^{NS}(x,t_0)\left(\frac{t}{t_0}\right)^{\frac{3}{2}A_f A(x)} \tag{14a}$$

Similarly the x- evolution of non singlet structure function will be

$$F_2^{NS}(x,t) = F_2^{NS}(x_0,t)\exp\left[-\int_{x_0}^{x}\frac{A(x)}{B(x)}dx\right]. \tag{14b}$$

In the NLO, the t and x evolution of non singlet structure functions will be obtain as

$$F_2^{NS}(x,t) = F_2^{NS}(x,t_0)\left(\frac{t}{t_0}\right)^{\frac{3}{2}A_f\left[A(x)+T_0 A_1(x)\right]} , \tag{15a}$$

$$F_2^{NS}(x,t) = F_2^{NS}(x_0,t)\exp\left[-\int_{x_0}^{x}\frac{A(x)+T_0A_1(x)}{B(x)+T_0B_1(x)}dx\right], \tag{15b}$$

with

$$A_1(x) = x\int_0^1 f(\omega)d\omega - \int_0^x f(\omega)\,d\omega + \frac{4}{3}N_f\int_x^1 F_{qq}(\omega)\,d\omega\ , \tag{16a}$$

$$B_1(x) = x\int_x^1\left[f(\omega) + \frac{4}{3}N_f F_{qg}^S(\omega)\right]\frac{1-\omega}{\omega}d\omega\ . \tag{16b}$$

Here we introduce an extra assumption $\left(\frac{\alpha_S(t)}{2\pi}\right)^2 = T_0\left(\frac{\alpha_S(t)}{2\pi}\right)$ [18], where T_0 is a numerical parameter. By a suitable choice of T_0 we can reduce the error to a minimum.

To compare our results with experimental data, we have to consider the relations between deuteron and proton structure functions measured in DIS with non-singlet quark distribution functions as

$$F_2^{NS}(x,t) = 3\{2F_2^p(x,t) - F_2^d(x,t)\}\ . \tag{17}$$

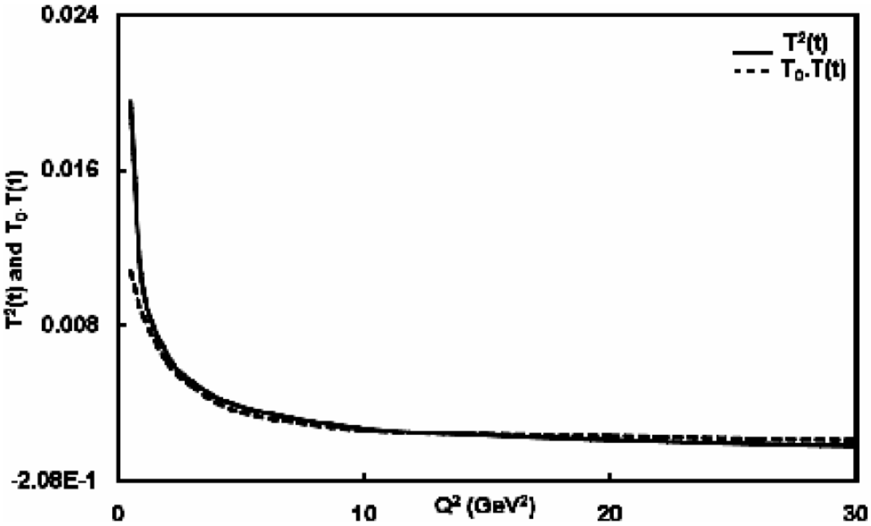

Figure 1. T^2 and $T_0.T$ curves.

3. Results and discussion

Here, we compare our result of *t* and *x* evolution of non-singlet structure function F_2^{NS} measured by E665 [19] (The data were taken at Fermilab experiment E665 in inelastic muon scattering with an average beam energy of 470 GeV) and NMC [20] (NMC in muon

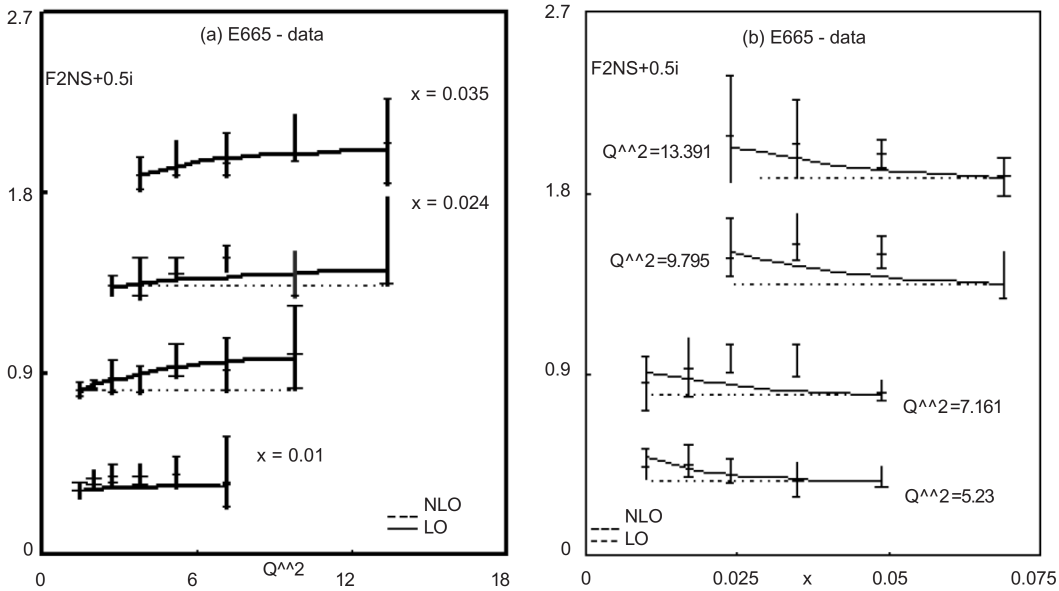

Figure 2 (a & b). Evolution of non-singlet structure function compared to E665 data.

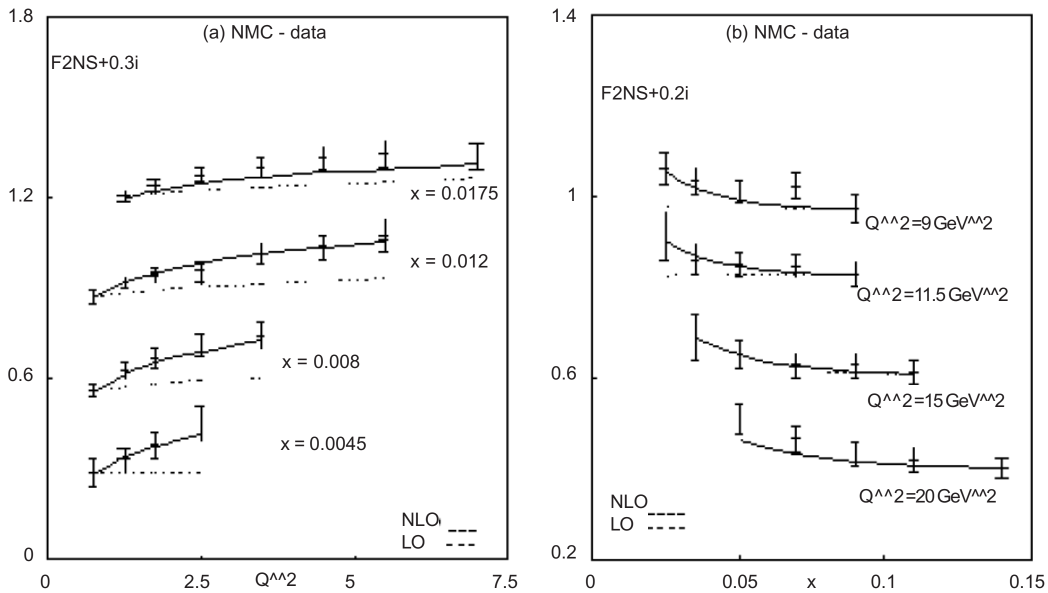

Figure 3 (a & b). Evolution of non-singlet structure function compared to NMC data.

deuteron DIS with incident momentum 90, 120, 200, 280 GeV). We consider the range of $0.01 \leq x \leq 0.0489$ and $1.496 \leq Q^2 \leq 13.391$ for E665, $0.0045 \leq x \leq 0.14$ and $0.75 \leq Q^2 \leq 20$ for NMC data. It is observed that within these range T_0 is satisfied for $0.08 \leq T_0 \leq 0.25$ (Figure 1). Figure 2(a) and Figure 3(a) represent the *t* evolution and Figure 2(b) and Figure 3(b) represent the *x* evolution of non-singlet structure function. For convenience, value of each data point is increased by adding 0.5i or 0.3i or 0.2i, where $i = 0, 1, 2, 3, \cdots$ are the numberings of curves counting from the bottom of the lowermost curve as the 0^{th} order. Here errors bars represent total combined statistical and systematic uncertainty. Our results are compatible with experimental values and fitness is better in NLO than the LO case.

4. Conclusion

Though there are various methods to solve DGLAP evolution equation to calculate quark and gluon structure functions, our method of characteristics to solve these equations is also a viable alternative. Though mathematically vigorous, it changes the integro-differential equations into ODE and then makes it possible to obtain unique solutions.

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